Besvarelser til Lineær Algebra
Ordinær Eksamen - 3. Januar 2017

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Bemærk, at der kan være sneget sig fejl ind.
Kontakt mig endelig, hvis du skulle falde over en sådan.
Dette dokument har udelukkende til opgave at forklare,
hvordan man kommer frem til facit i de enkelte opgaver.
Der er altså ikke afkrydsningsfelter, der er kun facit og tilhørende udregninger.
Udregningerne er meget udpenslede, så de fleste kan være med.

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Problem 1

What is the number of solutions of the following system of linear equations

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 + x_2 + x_3 &= 3 \\ x_1 + x_3 &= 0. \end{aligned}$$

Answer:

We can express the above system of linear equations using matrix notation:

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Reducing A to echelon form i.e finding all pivot entries is enough. We get:

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Since there is a pivot entry in the last column the system is inconsistent and therefore no solution exists.

Problem 2

Let A be a $4 \times n$-matrix and let E be the elementary matrix

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

How does the matrix EA appear from A?

Answer:

If one does not know how the changes of E affects A let $A = I_4$ where I_4 is the identity matrix. Of course it is easy to see that $EA = EI = E$ but the answer might be clearer if we write out these matrices:

$$EA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = E.$$

The change from A to EA is that we add -3 times row 1 to row 3 - which is the answer.

Problem 3

Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 3 & 4 \\ 1 & 1 & 0 & 2 & 2 \end{bmatrix}.$$

1. What is the value of n?

Since $T\colon \mathbb{R}^n \to \mathbb{R}^m$ we are asked "How many entries does a vector, multiplied on the right side of A, have?"This can be rephrased: "What is the number of columns of A?- Which should be 5 for the dimensions to fit.

2. What is the value of m?

Again since $T\colon \mathbb{R}^5 \to \mathbb{R}^m$ we are asked "How many entries does the resulting vector of the matrix product $A\mathbf{v}$ have?"which could be rephrased as "How many rows does A have?- Hence the answer is 3.

3. What is the rank of A?

We will reduce A to echelon form:

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 3 & 4 \\ 1 & 1 & 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

There is 3 out of 3 possible pivot entries hence the rank of A is 3.

4. What is the dimension of the null space of T?

We just have to remember that

$$\operatorname{rank}(A) + \operatorname{null}(A) = n \Leftrightarrow \operatorname{null}(A) = n - \operatorname{rank}(A),$$

hence we have that $\operatorname{null}(A) = 5 - 2 = 3$.

5. Is T one-to-one (injective)?

For T to be injective we must have pivot entries in each column which IS NOT the case. The answer is "No".

6. Is T onto (surjective)?

For T to be surjective we must have pivot entries in each row which IS the case. The answer is "Yes".

Problem 4

Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Then $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for $\mathbb{R}^3$.
Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be the orthogonal basis for $\mathbb{R}^3$ obtained by using the Gram-Schmidt process on $\mathcal{B}$. Then $\mathbf{v}_1 = \mathbf{u}_1$.
What is $\mathbf{v}_2$?

Answer:

By the Gram-Schmidt process, we have:

$$\begin{aligned}
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\mathbf{v}_1 \cdot \mathbf{u}_2}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \\
&= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1}{1^1 + 1^1 + 0^2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{2} \left(\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \\
&= \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}.
\end{aligned}$$

Problem 5

Let A be an $m \times n$ matrix, and let $B = [\mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3 \mathbf{b}_4 \mathbf{b}_5]$ and C be matrices satisfying that the products AB, BC and CA are defined.

How many columns are there in AB?

To get an overview let us list the things we know about A, B and C:

- A is $m \times n$.
- B consists of 5 column vectors. Hence B has 5 columns.
- Since AB is defined B must have n rows.
- B is therefore a $n \times 5$-matrix.

We now conclude, since AB is a $(m \times n)(n \times 5) = m \times 5$-matrix, that AB has 5 columns.

What is the size of BC?

We list more things we know to get an overview:

- Since CA is defined C has to have m columns.
- Since BC is defined C has to have 5 rows.

We conclude that BC is a $(n \times 5)(5 \times m) = n \times m$-matrix.

Problem 6

The characteristic polynomial of

$$A = \begin{bmatrix} -4 & 6 & -6 & 6 \\ -1 & 3 & -2 & 2 \\ -1 & 1 & 0 & 2 \\ -3 & 3 & -3 & 5 \end{bmatrix}$$

is $(t-1)(t+1)(t-2)^2$.

1. Let $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. For which value of λ is $A\mathbf{v} = \lambda\mathbf{v}$?

If we multiply A and $\mathbf{v}$ we get following:

$$A\mathbf{v} = \begin{bmatrix} -4 & 6 & -6 & 6 \\ -1 & 3 & -2 & 2 \\ -1 & 1 & 0 & 2 \\ -3 & 3 & -3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+0-6+6 \\ 0+0-2+2 \\ 0+0+0+2 \\ 0+0-3+5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

We see from the last equation that $A\mathbf{v} = 2\mathbf{v}$ hence $\lambda = 2$.

2. Which of the following is an eigenvector of A?

$$a) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad b) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad c) \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad d) \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

We have that $a)$ is the 0-vector - this can not be the answer because eigenvectors are defined to be all *non-zero* vectors $\mathbf{v}$ that satisfy the equation $A\mathbf{v} = \lambda\mathbf{v}$ for $\lambda \in \mathbb{R}$.

For $b)$ we see that this vector multiplied by A is:

$$A \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 & 6 & -6 & 6 \\ -1 & 3 & -2 & 2 \\ -1 & 1 & 0 & 2 \\ -3 & 3 & -3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \\ 0 \\ -3 \end{bmatrix}.$$

We can not find a scalar λ such that

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} -6 \\ -2 \\ 0 \\ -3 \end{bmatrix},$$

hence $b)$ is not an eigenvector.

For $c)$ we see that this vector multiplied by A is:

$$A\begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}-4 & 6 & -6 & 6\\-1 & 3 & -2 & 2\\-1 & 1 & 0 & 2\\-3 & 3 & -3 & 5\end{bmatrix}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}-4+6\\-1+3\\-1+1\\-3+3\end{bmatrix} = \begin{bmatrix}2\\2\\0\\0\end{bmatrix} = 2\begin{bmatrix}1\\1\\0\\0\end{bmatrix}.$$

We see, that $c)$ is an eigenvector by definition and it has the eigenvalue 2.

For $d)$ we have the same problem as with $b)$ because:

$$A\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \begin{bmatrix}-4 & 6 & -6 & 6\\-1 & 3 & -2 & 2\\-1 & 1 & 0 & 2\\-3 & 3 & -3 & 5\end{bmatrix}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \begin{bmatrix}-4-6\\-1-3\\-1-1\\-3-3\end{bmatrix} = \begin{bmatrix}-10\\-4\\-2\\-6\end{bmatrix}.$$

We can not find a scalar λ such that

$$\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \lambda\begin{bmatrix}-10\\-4\\-2\\-6\end{bmatrix},$$

hence $d)$ can not be an eigenvector.

3. Is A invertible?

Yes, because 0 is not one of the eigenvalues of A.

4. Is A diagonizable?

For A to be diagonizable it must have the same amount of eigenvectors as it has multiplicity of eigenvalues. We are guaranteed to have 1 eigenvector for each eigenvalue, so there is no problem for the eigenvalues 1 and -1 since their multiplicity is one. The eigenvalue 2 has a multiplicity of 2, but we have already concluded in the first and second part that

$$\begin{bmatrix}0\\0\\1\\1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}1\\1\\0\\0\end{bmatrix},$$

are eigenvectors with eigenvalue 2. Hence the amount of eigenvectors and the multiplicity of the eigenvalues are corresponding. We conclude that A is diagonizable.

Problem 7

Let T be the linear transformation with standard matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.
$\mathcal{B} = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \}$ is a basis for $\mathbb{R}^2$.
What is the matrix representation of T with respect to $\mathcal{B}$, denoted by $[T]_{\mathcal{B}}$?

Answer:

We are going to use the formula

$$[T]_{\mathcal{B}} = B^{-1}AB,$$

where $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. We have to find B^{-1}. Since $\det(B) = 1 \cdot 1 - 1 \cdot 0 = 1$ we have the inverse to be:

$$B^{-1} = \frac{1}{\det(B)} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

We can now use the formula to achieve the desired result:

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ 3 & 7 \end{bmatrix}.$$

Problem 8

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, let $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and let $\mathbf{u} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$.

1. Are the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ orthogonal?

Yes, because their dot-product equals 0:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 1 \cdot 1 + (-1) \cdot 1 + 0 \cdot 2 = 1 - 1 + 0 = 0.$$

2. What is the orthogonal projection of $\mathbf{u}$ on W?

We use the formula

$$\mathbf{w} = (\mathbf{u} \cdot \mathbf{v}_1)\mathbf{v}_1 + (\mathbf{u} \cdot \mathbf{v}_2)\mathbf{v}_2,$$

where $\mathbf{w}$ is the orthogonal projection. Please note that for the formula to work $\mathbf{v}_1$ and $\mathbf{v}_2$ must be orthonormal vectors. We have already concluded they are orthogonal, but we have to make their length 1. We find their current lengths:

$$\|\mathbf{v}_1\| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}, \qquad \|\mathbf{v}_2\| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}.$$

Since the lengths are not currently 1 we have to divide the vectors by their current length. We find the dot-products between $\mathbf{u}$ and the normalized versions of $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively:

$$\mathbf{u} \cdot \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \frac{1}{\sqrt{2}}((-2) \cdot 1 + 0 \cdot (-1) + 4 \cdot 0) = \frac{1}{\sqrt{2}}(-2) = -\sqrt{2}$$

$$\mathbf{u} \cdot \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \frac{1}{\sqrt{6}}((-2) \cdot 1 + 0 \cdot 1 + 4 \cdot 2) = \frac{1}{\sqrt{6}}(-2+8) = \frac{1}{\sqrt{6}}(6) = \sqrt{6}.$$

Hence by the formula we get using normalized $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{w} = -\sqrt{2}\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \sqrt{6}\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = - \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

3. What is the orthogonal projection of $\mathbf{u}$ on $W^\perp$?

We can simply use the formula

$$\mathbf{u} = \mathbf{w} + \mathbf{z} \Leftrightarrow \mathbf{z} = \mathbf{u} - \mathbf{z},$$

where $\mathbf{z}$ is the orthogonal projection of $\mathbf{u}$ on $W^\perp$, and $\mathbf{u}$ and $\mathbf{w}$ are the same as before. We get:

$$\mathbf{z} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}.$$

4. What is the dimension of $W^\perp$?

We know that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal and both lies in $\mathbb{R}^3$. This means that there is only 1 additional dimension which they do not account for. Hence the dimension of $W^\perp$ is 1.

Problem 9

Let $A = \begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 4 & 1 & -2 \\ 2 & 1 & 1 & -1 \end{bmatrix}$ and let $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$.

1. Is b contained in $\operatorname{Col} A$?

We need to see if we can make a linear combination of the vectors in A such that $\mathbf{b}$ is achieved, i.e. checking if the matrix $[A|\mathbf{b}]$ is consistent. We have:

$$\begin{bmatrix} 2 & 1 & 1 & -1 & 2 \\ 0 & 4 & 1 & -2 & 1 \\ 2 & 1 & 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 1 & -1 & 2 \\ 0 & 4 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}.$$

Since the last vector of the above matrix, $\mathbf{b}$, is a pivot column the system is not consistent. Hence $\mathbf{b}$ can not be written as a linear combination of the other vectors - therefore $\mathbf{b}$ is NOT in $\operatorname{Col} A$.

2. Is c contained in $\operatorname{Col} A$?

No, the dimensions of $\mathbf{c}$ does not match the size of the vectors in A.

3. Is b contained in $\operatorname{Null} A$?

No, the dimensions does not fit, since we require that

$$A\mathbf{x} = \mathbf{0}$$

for a vector in the null space of A.

4. Is c contained in $\operatorname{Null} A$?

For $\mathbf{c}$ to be in the null space of A we must have $A\mathbf{c} = \mathbf{0}$:

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 4 & 1 & -2 \\ 2 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 + 3 \cdot (-1) \\ 0 \cdot 0 + 1 \cdot 4 + 2 \cdot 1 + 3 \cdot (-2) \\ 0 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 + 3 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 0 + 1 + 2 - 3 \\ 0 + 4 + 2 - 6 \\ 0 + 1 + 2 - 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence $\mathbf{c}$ lies in $\operatorname{Null} A$.

Problem 10

Let A and B be 7×7 matrices with $\det A = 5$ and $\det B = 3$.

1. What is $\det(-A)$?

Since we A is a 7×7 matrix, we have:

$$\det(-A) = (-1)^7 \det(A) = -1 \det(A) = -1 \cdot 5 = -5.$$

2. What is $\det A^\top B$?

Using some properties of the determinant we get

$$\det A^\top B = \det(A^\top) \det(B) = \det(A) \det(B) = 5 \cdot 3 = 15.$$

3. What is $\det A^{-1}B$?

Using some properties of the determinant we get:

$$\det A^{-1}B = \det(A^{-1}) \det(B) = \det(A)^{-1} \det(B) = 5^{-1} \cdot 3 = \frac{1}{5} \cdot 3 = \frac{3}{5}.$$

Problem 11

Let $A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -1 & 1 & 1 \\ 0 & -2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 0 & 2 \\ 1 & 1 & 3 \\ 3 & 2 & 4 \end{bmatrix}$.
Let $C = AB$. What is the $(2,1)$-entry in C, i.e., c_{21}?

Answer:

By the definition of the matrix product we just have to multiply the second row of A with the first column of B:

$$\begin{bmatrix} 2 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \\ 3 \end{bmatrix} = 2 \cdot 2 - 1 \cdot 3 + 1 \cdot 1 + 1 \cdot 3 = 4 - 3 + 1 + 3 = 5.$$

Problem 12

Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 5 \end{bmatrix}$. What is the determinant of A?

Answer:

I will show two ways of calculating this. First the cofactor method around the first column:

$$\begin{aligned} \det(A) &= (-1)^{1+1} \cdot 1 \cdot \det \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} + (-1)^{2+1} \cdot 0 \cdot \det \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix} + (-1)^{3+1} \cdot 1 \cdot \det \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \\ &= \det \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} + \det \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \\ &= (2 \cdot 5 - 1 \cdot 1) + (1 \cdot 1 - 3 \cdot 2) \\ &= 9 - 5 \\ &= 4. \end{aligned}$$

Did we just achieve the answer? I think we did (Actually I did, you are just reading this you lazy bastard)! Can we use another method? You bet!

Consider A and let us reduce this to an upper triangular matrix:

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Since we only added a multiplum of one row to another we shall not multiply the determinant of the upper triangular matrix with anything. Since we have reduced it to a triangular matrix we can just multiply each diagonal entry. We get:

$$\det \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = 1 \cdot 2 \cdot 2 = 4.$$

Problem 13

Let $Q = c \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$, where a and c are constants. For which combination of a and c is Q an orthogonal matrix?

Answer:

We have two conditions for a matrix to be orthogonal. Each vector should be orthogonal, meaning their dot product should be 0. No matter which combination of vectors we choose for our dot product we get the same result, because of how the matrix Q is designed. We get:

$$c \begin{bmatrix} a \\ 2 \\ 2 \end{bmatrix} \cdot c \begin{bmatrix} 2 \\ a \\ 2 \end{bmatrix} = c^2(2a + 2a + 4) = c^2(4a + 4) = 0.$$

We can now just divide by c^2 and solve for a:

$$4a + 4 = 0 \Leftrightarrow a = -1.$$

We have that a must be -1 for this to work. Our current Q now looks like:

$$Q = c \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

The second condition is that all columns of the matrix must have length 1. As Q is all the columns has the same length which means, we can just use one of the columns to find c:

$$\sqrt{(-1c)^2 + (2c)^2 + (2c)^2} = 1.$$

The c is there because we must multiply the columns of Q by c. The equation above can be squared on both sides leaving:

$$(-1c)^2 + (2c)^2 + (2c)^2 = c^2 + 4c^2 + 4c^2 = 9c^2 = 1 \Leftrightarrow c^2 = \frac{1}{9} \Rightarrow c = \pm\frac{1}{3}.$$

Because of the quadratic part we have to include both positive and negative results - but only one of these answers are included in the exam answers. Hence we choose $a = -1$ and $c = \frac{1}{3}$ - however $a = -1$ and $c = -\frac{1}{3}$ would have been an equally correct answer (if it had been on the exam paper).

Problem 14

Let A be a 12×15 matrix.
Answer the following true/false problems about A.

1. A is a square matrix.

False, since $12 \neq 15$.

2. $\operatorname{Col} A$ is a subspace of $\mathbb{R}^{12}$.

True, because the vectors of $\operatorname{Col} A$ lies in the dimension equal to the amount of rows in A.

3. $\operatorname{Col} A$ is a subspace of $\mathbb{R}^{15}$.

False, because there is only 12 rows of A.

4. $\operatorname{Col} A$ and $\operatorname{Row} A$ have the same dimension.

True - the dimension is purely determined by the span of the vectors i.e., how many pivot entries there is in the columns and rows. Since there is an equal amount of pivot entries for both $\operatorname{Col} A$ and $\operatorname{Row} A$ the dimensions must be equal.

5. $\operatorname{Col} A$ and $\operatorname{Null} A$ have the same dimension.

False. $\operatorname{Col} A$ has dimension equal to the amount of pivot columns and $\operatorname{Null} A$ has the dimension equal to amount of columns in A minus the amount of pivot columns in A. Since there is an uneven amount of columns, 15, this can not be true.

Problem 15

The following commands are entered in the MATLAB Command Window:
» u = [1; 1; 1; 1];
» v = [1; 2; 3; 4];
» w = [1; 3; 6; 10];
» x = [1; 4; 10; 19];
» A = [u v w x];
» rref(A)
ans =

1	0	0	1
0	1	0	-3
0	0	1	3
0	0	0	0

» det(A)

1. Which one is true (see exam for options)?

v is a column vector because the semicolon makes the vector jump to a new row.

2. What is MATLAB's answer to the last command?

rref(A) is an upper triangular matrix with 0 on its diagonal, we get the determinant to be 0, because we have to multiply each diagonal entry.